JEE Physics Book self written

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Chapter 1

How to do your calculations

Here are some tips regarding how you do your calculations. Ideally you should have already thought about things like this and would be following the tips mentioned. These should be strong habits formed early on.

1. Take values common wherever you can (except with simple values where it doesn't really matter). It improves speed of calculation.

   Addition is generally faster than multiplication and taking values common reduces the number of multiplication steps you have to do. For example, suppose I obtain the equation

   $$\frac{1}{2}mv_1^2 + \frac{GMm}{r_1} = U_0 + \frac{1}{2}mv_2^2 + \frac{GMm}{r_2}$$

   All the values are known except for r_2, we need to simply plug in the values to obtain the answer.

   Bad way to solve:

   $$\frac{1}{2}mv_1^2 + \frac{GMm}{r_1} = U_0 + \frac{1}{2}mv_2^2 + \frac{GMm}{r_2}$$

   $$\frac{1}{2} \times 11 \times (1.44)^2 + \frac{6.67 \times 10^{-11} \times 2000 \times 11}{1.3 \times 10^{-9}}$$

   $$= 13.6 + \frac{1}{2} \times 11 \times (1.96)^2 + \frac{6.67 \times 10^{-11} \times 2000 \times 11}{r_2}$$

   $$11.405 + (11.287 \times 10^{-2}) = 13.6 + 21.13 + \frac{14.67 \times 10^{-7}}{r_2}$$

   Apart from saving time on calculations, taking values common is also arguably neater and saves time consumed in writing the same value repeatedly.

- If a value/variable is common across all or almost all terms, divide the entire equation by that value/variable (unless one of the values being divided is an unknown to be solved for).

  In this case m is a value common across almost the entire equation. We can divide to obtain

$$\frac{v_1^2}{2} + \frac{GM}{r_1} = \frac{U_0}{m} + \frac{v_2^2}{2} + \frac{GM}{r_2}$$

- Take variables common before substituting the values if possible

  We could have first substituted the values and then taken $m = 11$ common like (not recommended):

$$\frac{1}{2} \times 11 \times (1.44)^2 + \frac{6.67 \times 10^{-11} \times 2000 \times 11}{1.3 \times 10^{-9}}$$
$$= 13.6 + \frac{1}{2} \times 11 \times (1.96)^2 + \frac{6.67 \times 10^{-11} \times 2000 \times 11}{r_2}$$
$$\frac{1}{2} \times (1.44)^2 + \frac{6.67 \times 10^{-11} \times 2000}{1.3 \times 10^{-9}}$$
$$= \frac{13.6}{11} + \frac{1}{2} \times (1.96)^2 + \frac{6.67 \times 10^{-11} \times 2000}{r_2}$$

  You should always prefer taking values common before substituting values as it looks neater and it is easier to keep mental track of what you have written. It's also easier to identify values that can be taken common when they are variables.

- If a value/variable is common across some terms, just take it common (no use of dividing the entire equation)

  Now that we have divided the equation by m, we can still take GM commmon. We can also take constants such as $\frac{1}{2}$ common, though for simple constants it doesn't matter much.

$$\frac{v_1^2 - v_2^2}{2} + GM(\frac{1}{r_1} - \frac{1}{r_2}) = \frac{U_0}{m}$$

  At this point our equation is as simplified as possible and we can substitute values.

2. As you become comfortable doing calculations mentally, try reducing the number of steps you're actually writing down.

   Try doing two to three steps at a time mentally without writing down each individual step. This saves time in writing. If you are struggling a bit, practise. If you are struggling a lot, never mind, go back to how much you were writing earlier.

If you are able to cut down on steps but are making a lot more careless mistakes than earlier, practise reducing them by thinking through each step slowly. Again, if you are simply unable to avoid making lots of mistakes, go back to how you used to calculate earlier.

3. Avoid careless mistakes.

   Making more than one careless mistake in 10 calculation problems will significantly impact your jee results.

   - Make a conscious effort to reduce this.

     Slow down your speed to reduce mistakes, then try increasing speed again once you're comfortable at a slower speed.

   - If you tend to make careless mistakes, also make a habit of checking the approximate range of your answers (including intermediate steps).

     For instance, 23.45×487.61 is approximately $20 \times 500 = 10,000$. Check that your answer is approximately $10,000$. Once your accuracy is near perfect however, this may be an unnecessary step.

4. Get used to a certain kind of pen

   This could be either pens with a thick body or pens with a thin body. It could be a gel pen or a ball pen. Keeping your pen type consistent (ideally same brand) will improve writing speed as well prevent your hand from tiring out. JEE usually provides a thin body ball pen.

5. Diagrams should roughly indicate dimensional information

   Drawing diagrams exactly to scale is usually an unnecessary waste of time.

   - However, if you know the dimensions of the figure, drawing them approximately correctly relative to each other can be crucial in avoiding confusion later on.

     For instance a rectangle of 20cm $\times$ 67cm should have the longer side atleast twice as large as the smaller side. As another example, if you have to find the centre of mass of a 5kg mass and a 2kg mass, mark the centre of mass closer to the larger mass even though you haven't solved for the exact centre of mass yet.

     If you make a guess on a dimension while drawing a figure and it turns out vastly incorrect on solving, redraw the figure. For instance, given a set of masses if you mark the centre of mass to right of one of the masses but it turns out it actually lies to the left of it.

   - Sometimes you may avoid dimensional information altogether for convenience, but make a mental note of the fact that your diagram does not have dimensional information and then proceed.

     For example, when dealing with coordinate geometry in maths, it is common to not plot the points given or find the relative distances given since the solution of such problems often doesn't require either of these. In such a case, make all distances roughly equal; do not arbitrarily decide to make some distances longer than others.

If at any point you are confused or require better clarity of the problem to solve it, make a figure.

- If dimensional information is unknown, make all distances roughly equal in your figure, as in 3b

  This is common when dealing with generalised variables rather than values, or variables whose values are yet to be solved. Once you have solved values however, if the problem is not yet solved, you may redraw the figure. This is particularly important if the solved values are vastly apart, for instance if two distances that you drew as roughly equal turn out to be in the ratio of $1 : 100$

6. When to write units

   - Avoid writing units in intermediate steps if all your quantities are in SI
   - If all quantities can be easily converted to SI by multiplying/dividing by some power of 10, do so in the very first step or whenever your introduce those values into the calculation. Once converted, you can again avoid writing units repeatedly
   - When your units cannot be converted to SI instantly, keep track of units. Writing them repeatedly every step is still unnecessary, write them once somewhere on the page.
   - If you're using a well known set of units, such as bar-litre for thermodynamics or celsius for temperature, converting to SI is unnecessary.

Chapter 2

Trigonometry

You would have learnt basic trigonometry in class 10 maths. You will learn a lot more for jee, mainly in maths.

2.1 Angles

An angle measures the relative rotation between two rays with common origin. Two lines must exist in the same plane for there to exist an angle between them (because we need them to intersect), and this angle is measured in the common plane. For 2d geometry this is irrelevent, in 3d it matters.

2.2 Directed and undirected angles

2.2.1 Undirected angles

Angles can be directed or undirected. Angles you have studied so far are undirected and their range is $[0°, 180°)$. This is interval notation and denotes $0° \leq \theta < 180°$ (square bracket means $\leq$, $\geq$ while simple bracket means $<$,$>$). Angles above 180° (but below 360°) can be converteed to standard form by subtracting them from 360°. Angles above 360° can be converted to standard form by first adding/subtracting multiples of 360° to get them in the 0° to 360° range followed by subtracting from 360° if the angle is obtuse

Q. Convert undirected angle $-1643°$ to simplest form

A. First note that a negative sign has no relevance for undirected angles, as undirected angles are always positive. Delete the minus sign to get 1643°

Now subtract 360° repeatedly until we get the value below 360°. Subtracting $360° \times 4 = 1440°$ gives 203°. This is a reflex angle. Subtract it from 360° to obtain 157° which is the answer.

2.2.2 Directed angles

A directed angle measures the rotaton of a second ray from a first ray. Rotation can be clockwise or anti-clockwise, by convention anti-clockwise is positive and clockwise is negative.

(insert figure of +30 degrees and -30 degrees)

Standard domain of a directed angle is $[0°, 360°)$ or $(-180°, 180°]$, to convert an angle falling out of standard domain, repeatedly add or subtract $360°$

Q. Convert directed angle $-1643°$ to simplest form

A. Add $360°$ repeatedly. Adding 4 times gives $-3°$. You can leave the answer as such, or add $360°$ again to obtain positive answer $357°$

When considering undirected angles, $+20°$ is the same as $-20°$ which is the same as $340°$ which is the same as $-340°$. An undirected angle does not indicate clockwise / anti-clockwise, it only indicates the relation between two rays.

A directed angle, however, allows you to uniquely define the position of a second ray with respect to a first ray. For instance, if I have a ray drawn and I tell you that a second ray makes angle $20°$ (undirected) with it. This gives you two possibilities for the second ray. However if if I specify $+20°$ or $-20°$, it uniquely specifies the second ray. So $+20°$ is the same as $-340°$ since they both specify the same ray, similarly $-20°$ and $+340°$ specify the same ray.

(insert appropriate figures)

In the $[0, 180°]$ range, both directed and undirected angles behave in exactly the same way.

2.2.3 Angle between lines

Note that the two methods measure an angle between two rays (line segments with a direction). What about the angle between two intersecting lines? Two intersecting lines form two pairs of angles, one pair being obtuse and the other pair being acute (boundary cases are perpendicular lines and coincident lines). We could define a convention that we will only look at the smaller angle; after all, the larger angle can simply be found from the smaller one. In such a case our angles get the range $[0°, 90°]$

(INSERT figures of acute/obtuse, perpendicular, coincident and parallel)

2.3 Radian system of measurement

$360°$ has been arbitrarily decided as the measure of a full rotation in the degree system of measurement. Isn't a measurement such as $\frac{1}{12}$ rotation more natural than $30°$? Yes, it is. The radian system is somewhat similar.

The radian system defines 1 full rotation as 2πrad. Here $\pi = 3.1415\ldots$, so

$$1\text{rad} = \frac{360°}{2\pi} = 57.296\ldots$$

The degree symbol is not used with radian measurements.

Defining 1 rotation as 2πrad might feel weird at first, here's the reasoning behind it.

Q. Given a circular arc that subtends θ° at the centre of a circle of radius r, find the length of the arc.

A. Simple unitary method

$$Angle = 360^\circ \implies l = 2\pi r$$

$$Angle = 1^\circ \implies l = \frac{2\pi r}{360}$$

$$Angle = \theta^\circ \implies l = \frac{2\pi r\theta}{360}$$

Q. Given a circular arc that subtends θrad at the centre of a circle of radius r, find the length of the arc.

A. Similarly

$$Angle = 2\pi\text{rad} \implies l = 2\pi r$$

$$Angle = 1\text{rad} \implies l = r$$

$$Angle = \theta\text{rad} \implies l = r\theta$$

The radian system of measurement allows for the very simple formula $l = r\theta$. This further allows for simple formulas such as $A = \frac{1}{2}r^2\theta$ for area of a sector or $v = r\omega$, $a = r\alpha$ for angular velocity and acceleration (will be taught later).

Memorise the following table, these are the most commonly used values:

Degrees	Radians
30°	$\frac{\pi}{6}$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$
180°	π
270°	$\frac{3\pi}{2}$
45°	$\frac{\pi}{4}$

2.4 Trigonometric functions

You would already be familiar with the following

(insert figure of right angled triangle with appropriate parts marked)

$$\sin\theta = \frac{Opp}{Hyp}$$

$$\cos\theta = \frac{Base}{Hyp}$$

$$\tan\theta = \frac{sin\theta}{cos\theta} = \frac{Opp}{Base}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

So far you would have studied this only for acute angles ($0° \leq \theta \leq 90°$). We will now learn how to define these for general angles.

A right angled triangle with an obtuse or reflex angle inside does not make sense, hence we will have to use a definition different from the one involving opposite, base and hypotenuse.

Assume a circle of radius 1 centred at the origin. If the ray OP makes directed angle θ with the positive x-axis, we define $\sin\theta$ to be the y coordinate of the point P and $\cos\theta$ to be the x coordinate of the point P. Then we define the other 4 functions in terms of sin and cos as earlier. So $\tan\theta = \frac{\sin}{\theta}\cos\theta$, $\sec\theta = \frac{1}{\cos}\theta$ and so on.

(insert appropriate figures)

As an example, a ray making angle $\frac{5\pi}{4}$ is shown in figure. Its coordinates are $(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$. So $\sin(\frac{5\pi}{4}) = x = \frac{-1}{\sqrt{2}}$, $\cos(\frac{5\pi}{4}) = y = \frac{-1}{\sqrt{2}}$, $\tan(\frac{5\pi}{4}) = \frac{x}{y} = 1$ and so on. As you can see, the trigonometric functions of $\frac{5\pi}{4}$rad share some relation with those of $\frac{\pi}{4}$rad due to symmetry

(insert appro. figures)

We can divide the x-y plane into 4 quadrants based on the sign of x and y coordinates in that plane. We can then also find out the sign (positive/negative) of each of the trigonometric functions in that quadrant.

Range for θ	x	y	Quadrant	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0 < \theta < \frac{\pi}{2}$	$+$	$+$	Ist	$+$	$+$	$+$
$\frac{\pi}{2} < \theta < \pi$	$-$	$+$	IInd	$+$	$-$	$-$
$\pi < \theta < \frac{3\pi}{2}$	$-$	$-$	IIIrd	$-$	$-$	$+$
$\frac{3\pi}{2} < \theta < 2\pi$	$+$	$-$	IVth	$-$	$+$	$-$

Note that trigonometry strictly uses directed angles, in fact this was the main motivation of teaching directed angles to begin with. Ofcourse it does not matter in the $(0, \pi)$ range, but once outside this range it is something to keep note of.

2.5 Formulae

(insert list of formulae here)

Proofs of these formulae will be covered in maths. For now just memorise them or come back to them later.

Extra box:

Simple proof for $\sin(A + B) = \sin A \cos b + \cos A \sin B$

https://commons.wikimedia.org/wiki/File:AngleAdditionDiagramSine.svg

2.6 Graphs of trigonometric functions

(insert graphs here)

You should remember the graphs of sin, cos and tan, and be able to plot the graphs of the other three using the first three if required.

2.7 Inverse trigonometry

INCOMPLETE. will come back to this later

Chapter 3

Functions and graph-plotting

(INSERT all the graphs and tables of sets of points used for this chapter)

Q. Plot $y = x^2 + 3$

A. Take a few points and plot the function

Q. Plot $y = 2 - x$

A. Take a few points and plot. This is a straight line, so two points will do.

Q. Plot $x^2 + y^2 = 1$

A. Here y is not a function of x, nevertheless we can plot it. Take some points that satisfy the equation and plot. Incidentally, we get the plot of a circle

Q. Plot $x^2 + y^2 = -1$

A. This equation has no solutions, hence no plot. The square of a number is always positive and positive numbers cannot add up to a negative number

Q. Plot $y = \sin x$

A. Plot

Q. Plot $y = \sin 2x$

A. Here we have a 2 attached to x. We can plot this by squeezing the previous plot by a factor of 2 along the x-axis

Q. Plot $y = \sin \frac{x}{2}$

A. Here we can stretch the graph by a factor of 2 along the x-axis

Q. Plot $y = 2 \sin x$

A. We will stretch the graph for $y = \sin x$ by a factor of 2 along the y-axis. Note that we can rewrite the equation as $\frac{y}{2} = \sin x$ qand the same rules that apply for x also apply for y

Q. Plot $x = \sin y$

A. We will simply flip the graph along the line $x = y$ since x and y have swapped roles. This is again to show that x and y are identical in all respects, it is just customary to plot a function of the form $y = f(x)$ where f satisfies the properties of a "function" (see ncert). But in general any relation between x and y can be plotted.

Q. Plot $y = \sin^2 x$

A. We will try squaring all the previous values of $\sin x$. Note that the negative portion of $y = \sin x$ has flipped over as negative values too become positive when squared. A positive increasing function remains increasing when squared and a positive decreasing function remains decreasing when squared.

Q. Plot $y^2 = sin^2 x$

A. Some thinking is required here. This equation basically translates to the union of the graphs of $y = \sin x$ and $y = -\sin x$

Q. Plot $y^2 = x^4 + x^2$

A. For positive x, it is clear to see that the function is strictly increasing. Increasing x will increase y. Also this is an example of an even function, meaning $f(x) = f(-x)$. Substituting $x = 2$ or $x = -2$ will provide the same value for y. We can simply plot the positive half of the function and mirror it to obtain the plot when x is negative.

(INCOMPLETE, possibly)

Chapter 4

Vectors

Q. A man travels 40 m east. Then 50 m north. Then 50 m south west (along the 45 degree line). Find the distance between his final and original position.

A. Man initally starts at coordinates $(0,0)$ suppose. Then goes to $(40,0)$. Then $(40,50)$. Then he travels along a 45 degree line, how do we solve now?

We could find his displacement along both x and y axes. By simple trigonometry, these displacements will be $50\sin 45^\circ$ and $50\cos 45^\circ$. Hence his final coordinates are $(40-\frac{50}{\sqrt{2}}, 50-\frac{50}{\sqrt{2}})$. We still need to find his distance from origin. We know that the distance of a point (x,y) from $(0,0)$ is simply $\sqrt{x^2+y^2}$ so we can apply this to get $\sqrt{(40-\frac{50}{\sqrt{2}})^2+(50-\frac{50}{\sqrt{2}})^2}$ which can then be somewhat simplified.

Why was it convenient to use coordinates for this problem? Couldn't we have just drawn a figure without x and y axes and then tried solving as a geometry problem?

When we have quantities that specify movement in arbitrary directions that need to be added (in this case, the various displacements of the person), often the easiest approach is to consider the influence of each of these movements along the coordinate axes. This provides the necessary motivation to define vectors as we soon will.

4.1 Vectors

A vector is a quantity that:

- has magnitude (magnitude is always positive)
- has direction
- obeys the vector law of addition (will be described soon)

Some examples of vectors would be position, displacement, velocity, force, etc.

In contrast, scalars are quantities that only possess magnitude and optionally a sign (+/-). For instance, work done is a scalar that can be both positive and negative, while distance is a scalar that is always positive.

A vector can be drawn on paper as an arrow. For instance a velocity of 5 m/s towards 45 degrees south of east can be drawn as an arrow with length 5 units pointing in the correct direction.

(INSERT figure)

A vector can also be denoted as a quantity with a small arrow above it, for instance $\vec{v}$. A vector has both magnitude and direction so we can denote this by $\vec{v} = v\hat{v}$. Here v is the magnitude and $\hat{v}$ is the direction. $\hat{v}$ is called a unit vector and its magnitude is 1 (meaning to say that the magnitude of the direction of $\hat{v}$ is 1 since all the magnitude of $\vec{v}$ is already stored in v).

It is common to specify vectors in terms of the x and y axes, so for instance we could have $\vec{v} = 4\hat{\imath} + 5\hat{\jmath}$. This indicates a vector whose x component is 4 and y component is 5 as shown in the figure.

(INSERT figure)

$\hat{\imath}$ is the unit vector in the x direction and $\hat{\jmath}$ is the unit vector in the y direction. When you encounter 3D geometry, $\hat{k}$ is the unit vector in the z direction.

The magnitude of $\vec{v}$, denoted by $|\vec{v}|$ or simply v is given by the square root of the sum of the squares of the individual components (as we already have somewhat seen). So $v = \sqrt{4^2 + 5^2} = \sqrt{41}$. The direction however will have to be calculated. We know that $\vec{v} = v\hat{v}$ so

$$\hat{v} = \frac{\vec{v}}{v} = \frac{4\hat{\imath} + 5\hat{\jmath}}{\sqrt{41}} = \frac{4}{\sqrt{41}}\hat{\imath} + \frac{5}{\sqrt{41}}\hat{\jmath}$$

This may look complicated but is the standard way of representing direction. Note that the magnitude of a direction (unit vector) is one, as seen by squaring and adding components.

$$|\hat{v}| = \sqrt{(\frac{4}{\sqrt{41}})^2 + (\frac{5}{\sqrt{41}})^2} = 1$$

4.2 Vector addition

Suppose we have a displacement vector of $2\hat{\imath} + 3\hat{\jmath}$. Then we have another displacement vector of $3\hat{\imath} + \hat{\jmath}$. What is the total displacement? Some thought will tell you that you can simply add the x and y components separately to obtain $(2 + 3)\hat{\imath} + (3 + 1)\hat{\jmath}$ which gives $5\hat{\imath} + 4\hat{\jmath}$

Now suppose these vectors were drawn instead of being written in component form (as $x\hat{\imath} + y\hat{\jmath}$). How do you add the two arrows given in the figure?

(INSERT figure with two arrows with space between them) (INSERT figure with the same two arrows placed tail to head)

It makes sense to place the vectors such that the head of one vector touches the tail of the next vector. The resultant vector is given by connecting the tail of the first vector to the head of the second vector. (Memorise this, it is important to know which head touches which tail)

(INSERT figure with resultant vector marked)

Vector addition is commutative, adding the first vector to the second will give the same result as adding the second vector to the first.

(INSERT two figures, one showing A + B and the other showing B + A)

There exists a relation between the magnitudes $|\vec{A}|$, $|\vec{B}|$ and $|\vec{A}+\vec{B}|$. Given the angle between the vectors to be θ, the vector law of addition states

$$|\vec{A}+\vec{B}|^2 = |\vec{A}|^2 + |\vec{B}|^2 + 2|\vec{A}||\vec{B}|\cos\theta$$

The angle between two vectors is measured as an undirected angle keeping the tails of both the vectors together. Note that undirected angle means it lies in range $[0°, 180°]$. Also we keep tails together, unlike when adding vectors when we place the vectors tail-to-head.

INSERT two figures, one head to tail and one tail to tail with theta marked in both

4.3 Scalar multiplication

A vector can be multiplied by a scalar to obtain another vector in the same/opposite direction. For instance $2\vec{A}$ has twice the magnitude and the same direction as A. $-1 \times \vec{A} = -\vec{A}$ has the same magnitude of $\vec{A}$ but the opposite direction.

(INSERT relevant figure)

This also provides the basis for vector subtraction. $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

To subtract $\vec{B}$ we first reverse its direction to obtain $-\vec{B}$, then add this to $\vec{A}$ vectorially.

Q. Prove distributivity of scalar multiplication over vector addition

$$k(\vec{A}+\vec{B}) = k\vec{A} + k\vec{B}$$

A. We could try using the graphical approach (draw vectors as arrows) or notation approach (write vectors in component form). In this case, we will be able to prove this very easily using graphical approach

(INSERT relevant figure)

4.4 Dot product

The previous operation involved taking a scalar and vector to produce a new vector. What about a multiplication that takes two vectors as input?

Two such operators exist, one is the dot product that returns a scalar and the other is the cross product that returns a vector.

The dot product is given by

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

where θ is the (undirected) angle between the vectors. Note that this means given fixed magnitudes A and B, the maximum value of $\vec{A}\cdot\vec{B}$ is AB, attained when $\theta = 0°$ (vectors are parallel whereas the minimum value of $\vec{A}\cdot\vec{B}$ is $-AB$, attained when $\theta = 180°$.

Q.

$$\vec{A} = 4\hat{i} - 2\hat{j} + 3\hat{k}$$

$$|\vec{B}| = 13$$

The angle between the vectors $\vec{A}$ and $\vec{B}$ is $\frac{\pi}{6}$ Find $\vec{A}\cdot\vec{B}$

A.

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

B and θ are known, we just need to find A

$$A = \sqrt{4^2 + (-2)^2 + 3^2} = \sqrt{29}$$

$$\vec{A}\cdot\vec{B} = \sqrt{29}\times 13\times\cos\frac{\pi}{6} = \frac{13\sqrt{29}}{2}$$

Q. Vectors $\vec{A}$ and $\vec{B}$ are shown in the figure. $B = 10$ is known. A is not given, however the component of $\vec{A}$ parallel to $\vec{B}$ is given, its magnitude is 5. Find $\vec{A}\cdot\vec{B}$ **(INSERT figure with A parallel marked)**

A.

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

Looking at the figure, we can figure out that the magnitude of the parallel component (given by length of the vector) is $A\cos\theta$ So $A\cos\theta = 5$

$$\vec{A}\cdot\vec{B} = AB\cos\theta = 10\times 5 = 50$$

This gives us a useful result:

$$\vec{A}\cdot\vec{B} = AB\cos\theta = AB_{\parallel} = A_{\parallel}B$$

Q. Is $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ always true?

A.

$$\vec{A}\cdot\vec{B} = AB\cos\theta_1$$

$$\vec{A}\cdot\vec{B} = BA\cos\theta_2$$

Clearly $\theta_1 = \theta_2$ since they both refer to the same angle Yes, dot product is commutative.

Q. What about $\vec{A}\cdot(\vec{B}\cdot\vec{C}) = (\vec{A}\cdot\vec{B})\cdot\vec{C}$

A. Dot product of two vectors is a scalar. Taking a dot product of this scalar with a third vector is meaningless, since dot product is an operation that only takes two vectors as input.

If the question was $\vec{A}(\vec{B}\cdot\vec{C}) = (\vec{A}\cdot\vec{B})\vec{C}$, then this is a false result because this is equivalent to $k_1\vec{A} = k_2\vec{B}$ where $k_1 = \vec{B}\cdot\vec{C}$ and $k_2 = \vec{A}\cdot\vec{B}$ are scalars. This will not hold true if $\vec{A}$ and $\vec{B}$ are in different directions.

Distributivity of dot product over addition (and subtraction)

$$(\vec{A}+\vec{B})\cdot\vec{C} = \vec{A}\cdot\vec{C} + \vec{B}\cdot\vec{C}$$

Proving it using the $AB\cos\theta$ will require a result from 3d geometry, so we will not prove it here.

Q. Find $(2\hat{\imath}+3\hat{\jmath}+4\hat{k})\cdot(-\hat{\imath}-\hat{\jmath}-2\hat{k})$

A. We can now solve this using distributivity law repeatedly.

$$(2\hat{\imath}+3\hat{\jmath}+4\hat{k})\cdot(-\hat{\imath}-\hat{\jmath}-2\hat{k})$$

$$= 2\hat{\imath}\cdot(-\hat{\imath}-\hat{\jmath}-2\hat{k}) + 3\hat{\jmath}\cdot(-\hat{\imath}-\hat{\jmath}-2\hat{k}) - 2\hat{k}\cdot(-\hat{\imath}-\hat{\jmath}-2\hat{k})$$

$$= (-2\hat{\imath}\cdot\hat{\imath} - 2\hat{\imath}\cdot\hat{\jmath} + 4\hat{\imath}\cdot\hat{k}) + (-3\hat{\jmath}\cdot\hat{\imath} + \ldots) + (-2\hat{k}\cdot\hat{\imath} + \ldots)$$

We just need to figure out the products $\hat{\imath}\cdot\hat{\imath}$, $\hat{\imath}\cdot\hat{\jmath}$, and so on.

Some thinking will tell you $\hat{\imath}\cdot\hat{\imath} = |\hat{\imath}||\hat{\imath}|\cos\theta = 1\times1\times\cos 0 = 1$

Similarly $\hat{\imath}\cdot\hat{\jmath} = |\hat{\imath}||\hat{\jmath}|\cos\theta = 1\times1\times\cos\frac{\pi}{2} = 0$

In general, multiplying a unit vector with itself will give 1 whereas multiplying two perpendicular unit vectors will give 0. This means out of the 9 terms we had, 6 will be 0 and 3 will be 1

So the answer is $-2\hat{\imath}\cdot\hat{\imath} - 3\hat{\jmath}\cdot\hat{\jmath} + 4\hat{k}\cdot\hat{k} = -2-3+4 = -1$

In general this gives us a very simple formula for dot product,

$$(a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k})\cdot(b_1\hat{\imath}+b_2\hat{\jmath}+b_3\hat{k}) = a_1b_1 + a_2b_2 + a_3b_3$$

4.5 Cross product

(Bored of this, INCOMPLETE)

4.6 Which quantities are vectors?

This section will use terms defined later in the 11th and 12th syllabi, so you could skip it and star-mark it for later.

Remember, a quantity can be represented by a vector if it has magnitude, direction and follows the vector law of addition.

Q. Is current a vector?

A. Current follows the kirchoff's law of addition. Given currents I_1 and I_2 entering a node, and I_3 exiting a node, we have $I_3 = I_1 + I_2$. The quantity of resultant current I_3 is independent of the direction of the incoming current. As such it makes sense to represent current using a signed scalar (sign indicates forward or backward flow) rather than 3 components.

(INSERT figure)

Q. Is $\frac{d\vec{a}}{db}$ a vector?

A.

$$\frac{d\vec{a}}{db} = \lim_{\Delta b \to 0} \frac{\Delta \vec{a}}{\Delta b} = \lim_{b_2 \to b_1} \frac{\vec{a}_2 - \vec{a}_1}{b_2 - b_1}$$

$\vec{a}_2 - \vec{a}_1$ is a vector, $b_2 - b_1$ is a scalar. Dividing a vector by a scalar gives a vector. Applying a limit does not change the vector/scalar property of a quantity. (Why?)

Hence, yes, $\frac{d\vec{a}}{db}$ is a vector

Q. If $\frac{d}{db}$ is replaced by $\frac{\partial}{\partial b}$ in the previous question?

A. Yes, still a vector. The way you take a limit does not change the vector nature nature or dimensions of the quantity, just like dividing a distance by time will always give a quantity with dimensions of speed, no matter how big or small the distance or time interval is.

As a rule of thumb, Δa and ∂a have the same dimensions and vector/scalar nature as a since they are just a subtraction between two close values of a

Q. To a directed angle θ, we associate a vector $\theta\hat{n}$ where the direction of $\hat{n}$ is perpendicular to the plane of the angle and is given by taking the cross product of the two rays that formed the angle.

As per this definition, is angle a vector?

A. If all angles are in the same plane, angles add and subtract nicely, giving the impression that angles might be vectors. However we should be able to add angles in different planes also. (Angles in same plane have parallel/anti-parallel $\hat{n}$, angles in different planes have different $\hat{n}$)

(INSERT figure with three rays in 3d emerging from a common point with pairwise angles marked)

In general, 3d geometry shows that angles cannot be added in 3d. As a special example, take this:

(INSERT figure similar to previous with all 3 angles 90)

Adding two 90° angles with perpendicular $\hat{n}$ gives another 90° angle in a third, perpendicular direction. This is similar to saying $\hat{i} + \hat{j} = \hat{k}$ and cannot be justified vectorially.

Q. What about $\partial\Theta$? What about $\omega = \frac{d\Theta}{dt}$

A. Infinitesimal angles can indeed be added vectorially as long as the result is still an infinitesimal (Proof not mentioned).

Hence $\partial\Theta$ and $\vec{\omega} = \frac{d\Theta}{dt}$ can be considered as vectors when needed.

Q. Is surface area a vector? As usual we define $\vec{A} = A\hat{n}$ where $\hat{n}$ is perpendicular to the area.

A.

First thing to consider is that areas can be on flat surfaces or curved (non-flat) surfaces.

(INSERT figure)

The sum of two non-parallel flat surfaces will essentially give a non-flat surface.

(INSERT figure of square + square giving a new shape with two joint squares in different planes)

There is no obvious way of converting this non-flat surface area into a flat area. We can take its projection on a plane (as shown in figure) or we can find its magnitude (which is simply the sum of individual areas) but we can't represent the entire surface area in a more simple form. As such we will have to assume this weird shape as the simplest representation.

(INSERT projection)

Now coming to the question of whether area should be considered as a vector, our litmus test is the vector law of addition. If the vector formed by vector addition has real life significance and would represent what we would otherwise imagine to be the adddition two areas, then area is a vector.

We will take the following two flat areas that share a common edge of length h. The areas are $a \times h$ and $b \times h$ and they make an angle θ with each other.

(INSERT figure of triangle with coords (-x1,0), (0,h) and (x2,0) with relevant features marked)

Assuming axes as shown in the figure,

$$\vec{A}_1 = A_1 \hat{A}_1$$
$$A_1 = ah$$
$$\hat{A}_1 = -\sin\theta_1 \hat{\imath} + \cos\theta_1 \hat{\jmath}$$

$$\vec{A}_2 = A_2 \hat{A}_2$$
$$A_2 = bh$$
$$\hat{A}_2 = \sin\theta_2 \hat{\imath} + \cos\theta_2 \hat{\jmath}$$

But

$$\sin\theta_1 = \frac{h}{a}$$
$$\cos\theta_1 = \frac{x_1}{a}$$
$$\sin\theta_2 = \frac{h}{b}$$
$$\cos\theta_2 = \frac{x_2}{b}$$

On solving we get

$$\vec{A}_1 + \vec{A}_2 = h(x_1 + x_2)\hat{\jmath}$$

which is nothing but the area vector of the rectangle obtained by stretching the third side of the triangle. Area vector does indeed seem useful.

In fact, in general we can prove that for any closed surface (imagine a sphere or a cube),

$$\sum \vec{A} = 0$$

If the surface is curved, we will split into infinitely many small areas, each having their own vector. The sum of these will then add to zero.

$$\oint d\vec{A} = 0$$

This is a very important result, and will be helpful in electrostatics.

The three surfaces of the triangle do not together form a closed surface in 3d, but we can close it using two identical triangles whose vectors add up to zero.

(INSERT relevant figure)

Every open surface can be closed by a complementary surface, and both of these area vectors will have same magnitude and opposite directions. Using this we can find the area vector of complicated surfaces easily.

(INSERT figure of two open surfaces forming a closed surface)

The only downside of this area representation is that it does not tell you the total surface area of your original area, and the magnitude of total area of a curved area usually has no simply intuitive picture.

We also have a similar result in 2 dimensions (not as useful) for closed loops

$$\oint dl\hat{n} = 0$$

(INSERT figure)

Q. Prove $\oint dl\hat{n} = 0$ using $\oint d\vec{A} = 0$

A. Fairly straightforward now that we've already seen an example of exactly this. The triangle example we took was pretty much a 2d example which we then stretched to 3d. Lastly we closed both sides to obtain a closed surface. We can do the same here.

Consider an arbitrary closed loop and stretch it by h to the third dimension. Close it using two parallel faces.

$$\oint d\vec{A} = \left(\oint d\vec{A}_{loop}\right) + \vec{A}_{face1} + \vec{A}_{face2}$$

$$\vec{A}_{face1} = A\hat{n}_1$$

$$\vec{A}_{face2} = -A\hat{n}_1$$

Each differential area element is nothing but a rectangle with dimensions $dl \times h$

$$d\vec{A}_{loop} = hdl\hat{n}$$

Putting it all together, our proof is complete.

(MORE EXAMPLES, maybe)

Chapter 5

Calculus

Calculus is the mathematical study of continuous change, basically how a change in one variable causes a change in another related variable. Calculus that you will study can largely be separated in differential calculus and integral calculus

5.1 Differential calculus

5.1.1 Introduction and basic rules

Q. Suppose we have the function $y = x^2$. At $(1, 1)$ we draw the tangent of the curve and it intersects the x axis. Find the coordinates of the point at which it intersects the x axis.

A. Without prior background, this is a hard problem to solve. Nevertheless it has been given with the express purpose of improving the way you think. Spend a couple of minutes if you like, before looking at the solution.

First question that comes to mind - what is a tangent? Tangent is a straight line that touches the curve at exactly one point (unlike secants which intersect at two points).

Does a point have a unique tangent? Yes, generally it does. One way to imagine this is to imagine a car travelling along the curve $y = x^2$. At every point the car is pointing in a certain direction, and this direction is continuously changing. The direction the car points when it is at the point $(1, 1)$ is the direction of the tangent.

Now how to solve this? Note that what is important is the direction of the tangent. If I can find the direction, then using slope-point form ($y - y_1 = m(x - x_1)$), I can find the equation of the line since I already know one point on the line (which is $(1, 1)$). Using this I can get the coordinates.

Now how is slope expressed? Slope is basically the rate at which y grows versus x for a line. If y grows very quickly with respect to x for a given line, we say the slope is high, otherwise it is low.

Coming back to the car analogy, let's try imagining which direction will the car point when it is at $(1, 1)$. It will point to the next point it has to reach.

What is the next point it has to reach? Though we can't define an exact next point, we can take an approximate next point to be $(1.0001, 1.0001^2)$. This is the key insight that will help solve the problem. Now we know the car is at point 1 looking at a nearby point 2 - what is the direction in which the car is pointing?

Since we have two points, we can find slope aS

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1.0001^2 - 1}{1.0001 - 1}$$

We can simplify this either manually, or by using the $a^2 - b^2$ factorisation identity. Anything will do, I'll just show the factorisation identity for practice since it can often simplify calculations

$$\frac{1.0001^2 - 1}{1.0001 - 1} = \frac{(1.0001 + 1)(1.0001 - 1)}{1.0001 - 1} = 2.0001$$

Our answer is slightly above 2. Note that our selection of 1.0001 as the x coordinate of the second point was arbitrary, let's try with $1.00000001 = 1+10^{-8}$ instead

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(1 + 10^{-8})^2 - 1}{1 + 10^{-8} - 1} = \frac{(1 + 10^{-8} + 1)(1 + 10^{-8} - 1)}{1 + 10^{-8} - 1} = 2 + 10^{-8}$$

Our answer is even closer to 2. This seems to indicate our answer might indeed be 2. A useful thing to do at this point would be to generalise to an arbitrary h as the increase in x, so our coordinates now become $(1, 1)$ and $(1 + h, (1 + h)^2)$

Do this yourself. On solving you will get $m = 2 + h$

Clearly as h goes to zero, m goes to 2. This is our answer and is represented as

$$\frac{dy}{dx} = 2$$

which can be simply read as "D Y by D X equals two".

This notation is also useful because it tells us exactly what our answer represents, a small change in y divided by a small change in x. In general, we use Δ to represent a change, and d to represent an infinitesimal change, so

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{y_2 - y_1}{x_2 - x_1}$$

This can be read as "(limit as delta X tends to 0) of (delta Y by delta X)"

This shows that derivative is nothing but **a ratio of infinitesimal changes**, something that should always be remembered.

Q. Find $\frac{dy}{dx}$ at the point $(2, 4)$ on the curve $y = x^2$

A. Try doing this yourself. Follow the exact same steps we used. You should get 4 as the answer.

Possible doubt: Should we take $(2.0001, 2.0001^2)$ or $(2.0002, 2.0002^2)$ as the second point?

Answer to doubt: It makes no difference, the ideal method is to anyway take $(2+h, (2+h)^2)$ as the second point. If you are uncomfortable you may feel like shying away from using h because I am yet to provide a solution using h, however this is deliberate. Being able to generalise a special case of a problem to a general case by adding variables is both easy and an essential skill to have in mathematics. Hence you should practice it without looking at the solution.

Here's the solution using h

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(2+h)^2 - 2^2)}{(2+h) - 2} = \lim_{h\to 0} 4 + h = 4 + 0 = 4$$

Q. Let's generalise. Find slope at point (a, a^2) on the curve $y = x^2$
A.

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(a+h)^2 - a^2)}{(a+h) - a} = \lim_{h\to 0} 2a + h = 2a + 0 = 2a$$

Q. Find slope at point (a, a^3) on the curve $y = x^3$
A. Here we can use the identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ to simplify

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(a+h)^3 - a^3)}{(a+h) - a} = \lim_{h\to 0} (3a^2 + 3ah + h^2)$$

We simply substitute $h = 0$ to obtain the answer $3a^2$
Q. Find slope of curve $y = x^n$ at point (a, a^n)
A.

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(a+h)^n - a^n)}{(a+h) - a} = \lim_{h\to 0} \frac{(a+h)^n - a^n}{h}$$

Now we seem stuck. We do not know of any general identity yet for n degree.

A useful result is the binomial theorem which you will study extensively later on. Here's some motivation for it:

$$(a+h)^2 = a^2 + 2ah + h^2$$
$$(a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$$
$$(a+h)^4 = a^4 + 4a^3h + 6a^2h^2 + 4ah^3 + h^4$$

When we do $\frac{(a+h)^n - a^n}{h}$, the subtraction of a^n cancels the first term in the expansion of $(a+h)^n$. The second term is of the form $na^?h^?$ (question marks indicate powers) which will get divided by h. Similarly all the terms will get divided by h

The only useful term is the second term. Why? Because the terms are strictly decreasing. We have $h << a$ which means reducing the power of a by 1 and putting h instead is reducing our term to a negligible value. For example, consider $2, 2h, 2h^2, \ldots$ The only useful term is 2 when we apply the limit $h \to 0$. So we only need to find the second term of the binomial expansion.

$$\lim_{h\to 0}\frac{(a+h)^2-a^2}{h}=\lim_{h\to 0}\frac{(a^2+2ah+h^2)-a^2}{h}=\lim_{h\to 0}2a+h=2a$$

$$\lim_{h\to 0}\frac{(a+h)^3-a^3}{h}=\lim_{h\to 0}\frac{(a^3+3a^2h+3ah^2+h^3)-a^3}{h}=\lim_{h\to 0}3a^2+h(\ldots)=3a^2$$

$$\lim_{h\to 0}\frac{(a+h)^4-a^4}{h}=\lim_{h\to 0}\frac{(a^4+4a^3h+6a^2h^2+\ldots)-a^4}{h}=\lim_{h\to 0}4a^3+h(\ldots)=4a^3$$

Some pattern matching will tell you that this term is nothing but $na^{n-1}h$. On dividing by h we will obtain the result na^{n-1} which is our first major result in calculus

$$\frac{d}{dx}x^n = nx^{n-1}$$

Instead of using another variable a, it is common to denote the derivative in terms of x itself. This also provides for a new meaning of the derivative function. So far we were considering derivative at a point, where our input was a function and a point, and our output was a value. Here, however, our input is a function and our output is also a function, a function to which if you input the point, you'll get a value as input. Hence we can say that the derivative function is an example of a function that returns a function as output.

An easy way of remembering the above formula is, first take the power of the original term and place it on the side, then take the power and reduce it by 1. For instance if we have to find the derivative of x^18, we place 18 on the side of x^{18-1} to obtain $18x^17$

While in maths we need to be careful about things such as whether a function is differentiable or invertible, it is common in physics to blindly assume that functions are:

- functions - meaning that for each x in domain (in physics, domain = region of interest), there exists one and exactly one $f(x)$. A general equation between multiple variables that may not satisfy this property is called a relation.
- continuous throughout the domain - meaning that a small change in x produces a small change in $f(x)$. This also means that our function can be plotted on paper without lifting our pen as the function doesn't "jump"
- differentiable throughout domain - meaning that the derivative at every point exists and is unique. This will be covered better in maths
- infinitely differentiable - meaning that the derivatives can be further differentiated to obtain rate of change of derivatives (we will come to this soon)
- invertible - meaning that if y is a function of x, then x is also a function of y.

These are not always good assumptions, a rigorous method of solving would test for all these before proceeding. Neverthless, when solving physics, we tend to get used to certain "nice" categories of functions that satisfy all the above properties. JEE physics rarely requires calculus on functions that aren't "nice", though you should be thorough with them anyways from a JEE maths point of view.

Go slow, if you haven't understood any of what we have done so far, go back and read it a couple more times before proceeding.

Q. Find $\frac{d}{dx}(x^4 - 2x^2 + 1)$

A. Start with the definition of derivative. Ratio of a small change in a function of x, which is $f(x)$, divided by the corresponding small change in x

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - (h)}$$

$$= \lim_{h \to 0} \frac{[(x+h)^4 - 2(x+h)^2 + 1] - [x^4 - 2x^2 + 1]}{h}$$

Rearranging terms,

$$= \lim_{h \to 0} \frac{[(x+h)^4 - x^4] - [2(x+h)^2 - 2x^2] + [1-1]}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h} - \lim_{h \to 0} \frac{2(x+h)^2 - 2x^2}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h} - 2 \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

But $\lim_{h \to 0} \frac{(x+h)^4 - x^4}{h} = \frac{d}{dx}x^4 = 4x^3$ And $\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \frac{d}{dx}x^2 = 2x$
Hence we get our result

$$= 4x^3 - 2(2x) = 4x^3 - 4x$$

Let's try another one for practice

(INSERT sample question and answer on polynomial differentiation)

We have observed a lot of useful results from this. First result:

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

If two functions are being added, we can find their individual derivatives and then add them to obtain the derivative of the original function.

Next result:

$$\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$$

If we have a constant (such as 2, 3, π) multiplied to a function, we can find the total derivative by multiplying the constant to the derivative of the inner function.

Another result:

$$\frac{d}{dx}c = 0$$

Derivative of a constant is zero (what is a constant again?). This makes sense as the derivative is the ratio of infinitesimal changes, and a change in the value of x will not result in a change in the value of $f(x)$ if $f(x)$ is a constant.

Q. Find the derivative of $\sin x$

A. We start with the definition of derivative.

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h}$$

We can now use the identity $\sin A + B = \sin A \cos B + \cos A \sin B$

$$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0}\sin x(\frac{1-\cos h}{h}) + \cos x(\frac{\sin h}{h})$$

We're stuck. If we use some properties of the theory of limits we can easily solve this bit. More specifically,

$$\lim_{h\to 0}\frac{1-\cos h}{h} = 0$$

$$\lim_{h\to 0}\frac{\sin h}{h} = 1$$

Substituting these gives

$$\frac{d}{dx}\sin x = \cos x$$

Just to reiterate, what does this mean? A derivative is a ratio of changes, so if we make a small change in the value of x and observe the corresponding small change in $\sin x$, the ratio of these changes is $\cos x$. It also means that the slope of the graph $y = \sin x$ at a point $(a, \sin a)$ will be $\cos a$, so $\tan\theta = \cos a$. This may by now be obvious to you, but if not, this is a friendly reminder.

But the proof must have felt like cheating, since we magically brought up some results to prove it. Let's try proving them.

Q. Prove $\lim_{h\to 0}\frac{\sin h}{h} = 1$

A. We need to prove that for small values of h, $\sin h$ becomes arbitrarily close to h. We can plug in smaller and smaller values of h and find that $\frac{\sin h}{h}$ does indeed get closer to 1. Try substituting the values $h = 0.1$, $h = 0.001$ and $h = 0.00000001$ in a calculator. Or if that doesn't please you, find these values by plotting angles on a unit circle (as per the definition of $\sin x$)

But this merely observation, does that count as a formal proof? Since we know that $\sin x$ is continuous and it doesn't change rapidly at $x = 0$ (the derivative of $\sin x$ is not infinite), we can indeed say the results hold true.

A more rigorous proof than this is out of the scope of this textbook.

As a rule of thumb, just remember that as $h \to 0$

$$\sin h \approx h$$

$$\cos h \approx 1 - \frac{h^2}{2}$$

Q. Prove that $\frac{d}{dx}\cos x = -\sin x$

A. Left as a exercise. The proof is almost identical to the proof of the derivative of $\sin x$, so take hints from there

Q. Find the derivatives of the following if possible:

(i) $e^{\pi}x^{\pi-e}$ (ii) $(\sin x)x^4$ (iii) x^{2x}

A.

(i) e^{π} is a constant, we can find the derivative of the inner function and multiply it with this constant. The inner function is $x^{\pi-e}$ What is the derivative of the inner function?

This is of the form x^n. Here n is not a natural number but it doesn't matter, the result we proved for natural number powers of x also holds for real number powers due to the binomial theorem for real number powers (will be taught later in maths). So we find nx^{n-1} which is $(\pi - e)x^{\pi-e-1}$

Multiplying them, we get $e^{\pi}(\pi - e)x^{\pi-e-1}$

(ii) One may feel like assuming $\sin x$ to be a constant and keeping it aside as we have done before. But no, $\sin x$ is not a constant, it is a function itself. We are yet to learn how to differentiate the product of two functions, once we do, then we will be able to solve this.

(iii) Does this fit the formula x^n? No because n must be a constant but we have a function in the power of x. There is a method to differentiate functions of the form $f(x)^{g(x)}$ but we will not cover it here

5.1.2 Product rule

How to differentiate the product of two functions?

More specifically, we want to find

$$\frac{d}{dx}f(x)g(x) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

We also know

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$g'(x) = \lim_{h\to 0}\frac{g(x+h) - g(x)}{h}$$

This means

$$f(x+h) \approx f(x) + hf'(x)$$

$$g(x+h) \approx f(x) + hg'(x)$$

Which makes sense because $f(x+h)$ should be equal to original $f(x)$ plus the change in $f(x)$. And the change in $f(x)$ is the ratio of changes (derivative) multiplied by the change in x (which is h)

Substitute this into our original problem

$$\frac{d}{dx}f(x)g(x) = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$
$$= \lim_{h\to 0}\frac{(f(x)+hf'(x))(g(x)+hg'(x))-f(x)g(x)}{h}$$

On opening the bracket and simplifying

$$= f'(x)g(x) + g'(x)f(x) + \lim_{h\to 0} hf'(x)g'(x)$$

The last term is simply zero due to leftover h (why? do you remember) and we get the result

$$(f(x)g(x))' = f'(x)g(x) + g'(x)f(x)$$

I've used quote mark (') to denote derivative as it is neater. To make it even neater, don't write x

$$(fg)' = fg' + gf'$$

How to remember:

- Differentiate the first function, multiply with the second function.
- Differentiate the second function, multiply with the first function.
- Add both to obtain the derivative of the product of the functions.

Q. Find $[(x^2+3x+2)(2x^2-x+14)]'$

A.

$$[(x^2+3x+2)(2x^2-x+14)]'$$
$$= [x^2+3x+2]'[2x^2-x+14] + [x^2+3x+2][2x^2-x+14]'$$
$$= [2x+3][2x^2-x+14] + [x^2+3x+2][4x-1]$$

Open brackets and simplify to obtain a cubic equation

We could have also opened brackets first to obtain a biquadratic polynomial and then differentiated. This was just a bit faster.

Q. Find $[(x^2+3x+2)(2\sin^2 x - \sin x + 14)]'$

A.

$$= [x^2+3x+2]'[2\sin^2 x - \sin x + 14] + [x^2+3x+2][2\sin^2 x - \sin x + 14]'$$

$$= [2x+3][2\sin^2 x - \sin x + 14] + [x^2+3x+2][2(\sin^2 x)' - \cos x]$$

Alas, we do not know how to differentiate $\sin^2 x$.

The answer is not $\cos^2 x$ if you're guessing that, we have not proved any general property that says $(f^2)' = (f')^2$

We can still apply a trick here to get the answer. $\sin^2 x = \sin x \sin x$

Now that we have the product of two functions whose derivatives we know, we can differentiate $\sin^2 x$

$$(\sin^2 x)' = (\sin x \sin x)' = \sin x(\sin x)' + (\sin x)' \sin x = 2 \sin x \cos x$$

We can substitute this back into our working to obtain the final answer.

5.1.3 Chain rule

Q. Find a function we do not know to differentiate yet

A. $\frac{\sin x}{x}$ is an example. We do not know how to find $(\frac{f}{g})'$ in terms of f' and g'

$\sin \sin x$ is another example. We do not know how to differentiate a function of a function

$$f(g(x)) = ?$$

The chain rule will address the second problem, and by doing so it will address the first one too.

Before we start with the result and proof, I hope you can see the difference between $fg = f(x) \times g(x)$ and $fog = f(g(x)$.

At this point, it becomes helpful to think of functions as something that take input and return output (because that's what they essentially are). $f(x)g(x)$ means taking a value x, inputting it to both f and g, getting the two distinct outputs and finding their product. $f(g(x))$ means taking a value x, inputting it to g, getting the output, sending that output as input to f, and then getting the output.

INSERT figure of functions as boxes with input and output arrows. Illustrate fg and fog

As a side note, multiplication, addition, subtraction, etc too are functions, just a different type of function you're not used to. So far you've dealt with functions of the form $f(x)$, meaning they take one input and give one input. These are called unary functions. Functions like addition take two inputs and return one input, and are called binary functions (not to be confused with the base-2 number system). We can write a general binary function as $f(x, y)$. Using such a notation, we can also write maths as follows

$$3 + 4 = +(3, 4) = 7$$

Here we have a function of the form $f(x, y)$. f is the addition operation $+$, and 3 and 4 are the inputs x and y.

Another example:

$\sin(x+3) + \sin x + 3 = +((\sin(x+3)), \sin(x), 3) = +((\sin(+(x,3))), \sin(x), 3)$

INSERT boxes figure

Here the outer addition is taking three inputs. How is that possible? We can add pairwise as follows

$$+(x_1, x_2, x_3) = +(+(x_1, x_2), x_3)$$
$$+(x_1, x_2, x_3, x_4, \dots x_n) = +(+(\dots + (+(+(x_1, x_2), x_3), x_4), \dots, x_n)$$

(INSERT boxes figure)

So to be very pedantic, we can write the above function as

$$+(+((\sin(+(x,3))), \sin(x)), 3)$$

INSERT boxes figure

Now that we are able to understand functions as boxes, and deeply understand what $f(g(x))$ means, let's try finding its derivative

$$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

We also know

$$[f(x)]' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$[g(x)]' = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

Using the following approximations (which we used when proving product rule) will make the proof easier (at the cost of rigour)

$$f(x+h) \approx f(x) + hf'(x)$$
$$g(x+h) \approx g(x) + hg'(x)$$

$$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$
$$\approx \lim_{h\to 0} \frac{f(g(x) + hg'(x)) - f(g(x))}{h}$$

Now inside f we have $g(x)$ which is a non-negligible quantity and $hg'(x+h)$ which is a small quantity with size of order h. Let's assume the following variables to make things simple.

$$X = g(x)$$
$$H = hg'(x)$$

So we have

$$\approx \lim_{h\to 0} \frac{f(X+H) - f(g(x))}{h}$$

$$\approx \lim_{h\to 0} \frac{f(X) + Hf'(X) - f(g(x))}{h}$$

Substituting back the values of X and H

$$\approx \lim_{h\to 0} \frac{f(g(x)) + hg'(x)f'(g(x)) - f(g(x))}{h}$$

$$= \lim_{h\to 0} \frac{hg'(x)f'(g(x))}{h}$$

$$= f'(g(x))g'(x)$$

This is our result. Learning to use this result is important and will come only with practice.

Q. Differentiate $\sin{(x^2)}$

A. We have a function of the form $f(g(x))$ where $f(x) = \sin x$ and $g(x) = x^2$

If you haven't understood yet, take a moment to understand why have we taken $f(x) = \sin x$ and not $f(x) = \sin x^2$

The reason is simply that if we take $f(x) = \sin x^2$, then our function which we have differentiate is of the form $f(x)$ and not $f(g(x))$ which is what we need. Our outer function is sin and our inner function is squaring. It also becomes useful to think of functions in terms of what they do and not what they "do" on. When we write $f(x) = \sin(x)$, x is just a dummy variables that has no actual meaning. It could just as well have been $f(y) = \sin y$ or $f(2a+b) = \sin{(2a+b)}$ or $f(g(x)) = \sin{(g(x)}$. The only thing important about f is that it takes the sin of whatever input it gets. Similarly g outputs the square of whatever input it gets.

The result should be $f'(g(x))g'(x)$

$$f'(x) = \cos x$$

By simple substitution

$$f'(g(x)) = \cos{(g(x))} = \cos{(x^2)}$$

$$g'(x) = 2x$$

Therefore

$$(\sin{(x^2)})' = 2x\cos{(x^2)}$$

Q. Find the following

(i) $(\sin 4x)'$ (ii) $(\sin^2 x)'$

A. (i)

$f(g(x)) = \sin 4x$ The obvious choices for f and g are

$$f(x) = \sin x$$

$$g(x) = 4x$$

$$f'(x) = \cos x$$

$$f'(g(x)) = \cos 4x$$

$$g'(x) = 4$$

$$(f(g(x)))' = f'(g(x))g'(x) = 4\cos x$$

(ii)

Note that here the outer function is squaring and the inner function is sin. DIY

Explicitly writing f and g every time is an effort, with practice we will learn how to write the derivative directly. A simple way to remember is "derivative of outer with inner as input. Then derivative of inner"

Q. Find $(\sin(\cos(\sin x)))'$

A. We start from the outermost function and work our way inwards. Our outermost function is sin, its derivative is cos. We now give whatever was inside as input (we had $(\cos(\sin x))$ inside) to obtain $\cos(\cos(\sin x))$

Then we multiply this with the derivative of what was inside

$$\cos(\cos(\sin x))(\cos(\sin x))'$$

We need to apply chain rule a second time to obtain the derivative of this second function. We focus only on this task, which is finding $(\cos(\sin x))'$. Now our outer function is cos whose derivative is $-\sin$. We again give whatever was inside as input (we had $\sin x$ inside) to obtain $-\sin(\sin x)$

Then we multiply this with this with the derivative of what was inside to obtain

$$\cos(\cos(\sin x))(-\sin(\sin x))(\sin x)'$$

Lastly we need the derivative of $\sin x$ which we already know. This gives the following as answer.

$$-\cos(\cos(\sin x)) \cdot \sin(\sin x) \cdot \cos x$$

Convince yourself that this is the simplest form in which this answer can be expressed. For instance, we can't merge any terms because composite functions $(f(g(x)))$ and multiplying functions $(f(x)g(x))$ are two different things.

5.1.4 Quotient rule

Q. Find $(\frac{f}{g})'$ in terms of f' and g'

A. We do not yet have a "division rule" - a way to differentiate functions composed using the division (/) operator. How to proceed?

We know to differentiate products of functions and functions with exponents. We can perform a division $(\frac{f}{g})$ using multiplication and a -1 exponent to find the inverse.

$$\frac{f}{g} = f \cdot g^{-1}$$

Now we can differentiate it

$$(\frac{f}{g})' = (f \cdot g^{-1})'$$

Outer function is multiplication

$$(\frac{f}{g})' = f \cdot (g^{-1})' + f' \cdot g^{-1}$$

f' is assumed to be known. $(g^{-1})'$ is unknown but can be simplified using our exponent rule $((x^n)' = nx^{(n-1)})$.

$$(g^{-1})' = -1 \cdot g^{-2} \cdot g'$$

Why have we multiplied g' at the end? We have done so because we are differentiating with respect to x, not g, and as per chain rule g is a function of x and still needs to be differentiated.

$$\frac{d(g^{-1})}{dg} = -g^{-2}$$

$$\frac{d(g^{-1})}{dx} = -g^{-2}g'$$

(where $g' = \frac{dg}{dx}$)

Putting it all together gives

$$(\frac{f}{g})' = f \cdot (g^{-1})' + f' \cdot g^{-1} = -\frac{fg'}{g^2} + \frac{f'}{g}$$

$$(\frac{f}{g})' = \frac{gf' - fg'}{g^2}$$

This is the "quotient rule" and can be memorised.

INSERT more solved examples of chain rule

By now you should be confident that, given a function in one variable (say x), if the following properties are followed, you should be able to differentiate it:

- It is composed of base functions such as polynomials, trigonometric functions (sin, cos, etc)
- The base functions are joined by the operators $+$, $-$, $*$, $/$, $\hat{}$ (with constant exponent, variable exponent will be taught later)
- The derivatives of all the base functions are known

5.1.5 Variables and equations

So far we have only differentiated explicit functions of x. Given f, we found $\frac{df}{dx}$ and called this f'

$$\frac{df}{dx} = f'$$

Or

$$\frac{df(x)}{dx} = f'(x)$$

To make things convenient, we can also use the following notation

$$df = f'dx$$

Or

$$df(x) = f'(x)dx$$

So we are simply saying that a small change in $f(x)$ is the product of the derivative and the small change in x. Seems pretty straight forward, but is it really? Also how is this useful?

Firstly, from a rigorous mathematical perspective, this is not straightforward because $df(x) = f'(x)dx$ does not mean $\lim_{\Delta x \to 0} \Delta f(x) = f'(x) \lim_{\Delta x \to 0} \Delta x$. If you have not done theory of limits, you may skip this bit. If you have, you would notice that the right hand side of the equation is always zero independent of the nature of f. Why? Because $\lim_{\Delta x \to 0} \Delta x = 0$ by the property of limits. Therefore, what we actually mean when we say $df(x) = f'(x)dx$ is actually $\frac{df(x)}{dx} = f'(x)$ and we are not actually allowed to take the denominator to the other side from a limits point of view, we do it only for convenience.

$$\frac{df(x)}{dx} = f'(x) \iff \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = f'(x) \iff df(x) = f'(x)dx$$
$$\not\Longleftarrow \lim_{\Delta x \to 0} \Delta f(x) = f'(x) \lim_{\Delta x \to 0} \Delta x$$

Now how is this useful? Why use just da or just db instead of always using both the ds together like $\frac{da}{db}$?

Because it makes calculations convenient when dealing with equations rather than functions. Here's an example

Q.

$$y^2 + 1 = x^3$$

Find $\frac{dy}{dx}$

A. First and most obvious method would be to write y explicitly as a function of x before differentiating. So we'll get

$$y = \sqrt{x^3 - 1}$$

Then we can differentiate and apply chain rule to get

$$\frac{dy}{dx} = \frac{1}{2}(x^3 - 1)^{-\frac{1}{2}} 3x^2 = \frac{-3x^2}{2\sqrt{x^3 - 1}}$$

That is indeed the answer. Here's another method of doing it that should also give the same result.

$$d(y^2 + 1) = d(x^3)$$

If $a = b$ then a small change in a should equal a small change in b

$$LHS = d(y^2 + 1) = 2ydy$$

(Explanation: $\frac{d(y^2+1)}{dy} = 2y$)

$$RHS = d(x^3) = 3x^2dx$$

(Explanation: $\frac{d(x^3)}{dx} = 3x^2$)

$$2ydy = 3x^2dx$$

$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

If needed, we can now find y explicitly in terms of x and substitute it in the denominator. Isn't this method neater?

If you're still not convinced, what about an equation where it is not possible to find one variable in terms of the other.

Q.

$$y^3 + 2y + 3 = x^3 - x$$

Find $\frac{dy}{dx}$ when $x = 2$

A.

$$d(y^3 + 2y + 3) = d(x^3 - x)$$

$$3y^2dy + 2dy + 0 = 3x^2dx - dx$$

$$(3y^2 + 2)dy = 3x^2dx$$

$$\frac{dy}{dx} = \frac{3x^2}{3y^2 + 2}$$

So to find $\frac{dy}{dx}$ at a given point, we can just plug in the values of x and y at that point. We are given $x = 2$, but we do not know the value of y. However we can find it using the value of x. Note that the point at which we are finding $\frac{dy}{dx}$ also lies on the curve hence the equation must hold true at that point.

$$y^3 + 2y + 3 = 2^3 - 2$$

$$y^3 + 2y - 3 = 0$$

We need to find the roots of this cubic equation. While there exists a formula for the same, it is lengthy and not worth memorising. Use hit and trial to obtain $y = 1$ as a solution, then factorise.

$$(y - 1)(y^2 + y + 3) = 0$$

Now to find the roots of $y^2 + y + 3$. we can use quadratic formula. $D = b^2 - 4ac = (1)^2 - 4(1)(3) < 0$. D is negative hence the roots are complex. In this instance we can neglect them, since JEE usually does not expect us to do calculus with complex roots.

We can also do this with multiple variables.

Q.

$$2a^3 = b^2 + c^4$$

Find $\frac{dc}{da}$ at (a=1,b=1) if it is known that $\frac{da}{db} = 4$ at this point

A.

$$d(2a^3) = d(b^2) + d(c^4)$$

$$6a^2 da = 2b db + 4c^3 dc$$

Now we are given that $\frac{da}{db} = 4$. Let's divide the equation by db to use this information

$$6a^2 \frac{da}{db} = 2b + 4c^3 \frac{dc}{db}$$

Now substitute the values at the known point.

$$6(1)^2(4) = 2(1) + 4c^3 \frac{dc}{db}$$

Let's find c at the given point $(a = 1, b = 1)$

$$2a^3 = b^2 + c^4$$

$$2(1)^3 = (1)^2 + c^4$$

$$c^4 = 1$$

$$c = 1, -1$$

Now substitute

$$6(1)^2(4) = 2(1) + 4c^3\frac{dc}{db}$$

$$\frac{dc}{db} = \pm\frac{22}{4}$$

Now we can use a trick to obtain the answer

$$\frac{dc}{da} = \frac{\frac{dc}{db}}{\frac{da}{db}} = \pm\frac{22}{16}$$

In a way, the differential elements can be treated separately and we can just cancel db and db to obtain the answer.

Here's an alternate method that would have worked out faster:

$$6a^2da = 2bdb + 4c^3dc$$

Divide this by da instead.

$$6a^2da = 2bdb + 4c^3dc$$

$$6a^2 = 2b\frac{db}{da} + 4c^3\frac{dc}{da}$$

$$6(1)^2 = \frac{2(1)}{\frac{da}{db}} + 4(\pm 1)^3\frac{dc}{da}$$

$$6 = \frac{2}{4} \pm 4\frac{dc}{da}$$

$$\frac{dc}{da} = \pm\frac{22}{16}$$

However